

System Design and Methodology / Embedded Systems Design

VII. Timed and Hybrid Automata

**TDTS07/TDDI08
VT 2026**

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(Based on material by Petru Eles and Soheil Samii)

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TIMED & HYBRID AUTOMATA

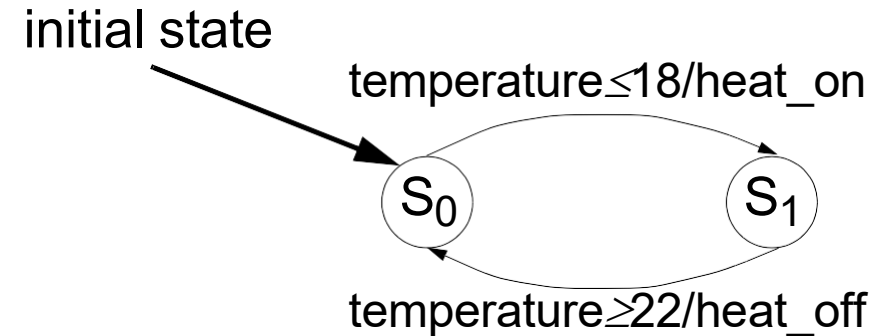
1. Restrictions of the synchronous FSM model
2. Timed automata
3. Timed automata as a particular case of hybrid automata
4. Hybrid automata

Restrictions/Assumptions with synchronous FSM

- FSMs react to inputs and generate, as response, outputs.
- The inputs are either present or absent; when inputs are present, the FSM reacts and generates outputs.
- Between the time instants when inputs are present, nothing interesting occurs; at each instant when inputs occur, a reaction (outputs) is computed *instantly* by the FSM.
- FSMs operate in a sequence of *discrete* reactions.
- The clock can be explicitly modelled as a FSM delivering ticks for the whole system; all transitions in the system are synchronised on this clock tick. Time (non-negative integer) is captured counting these clock ticks \Rightarrow *discrete time model*.

Example FSM: Thermostat

- Input event: {temperature}
 - Signal *temperature* is received at certain instants of time from a sensor;
 - When input *temperature* is received and the guard on the transition is true, the system generates an output.
- Outputs: {heat_on, heat_off}
- States: {S₀, S₁}
 - S₀: system cools (heating off)
 - S₁: system is heating



- We want to keep the temperature close to 20°;
- To avoid *chattering* (turning on and off rapidly, all the time), we allow temperature to be inside a band (technique called *hysteresis*).

Timed Automata

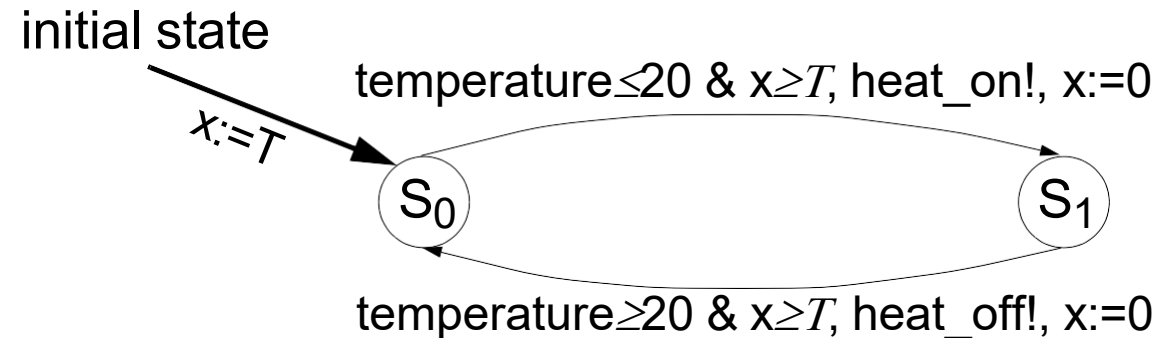
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Timed Automata

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- Real-time systems require measuring the passage of (continuous) time and performing actions at specific times.
- In timed automata time is considered a continuous quantity. No global synchronisation, in the sense of a unique clock, is assumed.
- Timed automata are an extension of the FSM model which allows modelling of certain real-time systems and formal reasoning about time.
 - A timed automaton is a finite automaton (similar to a FSM) *augmented with a finite set of real-valued clocks*.

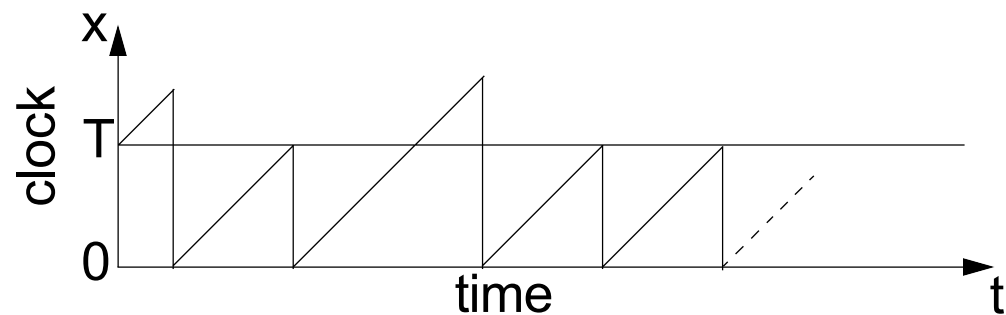
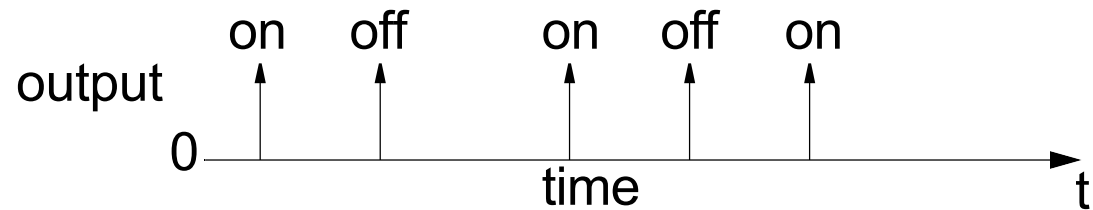
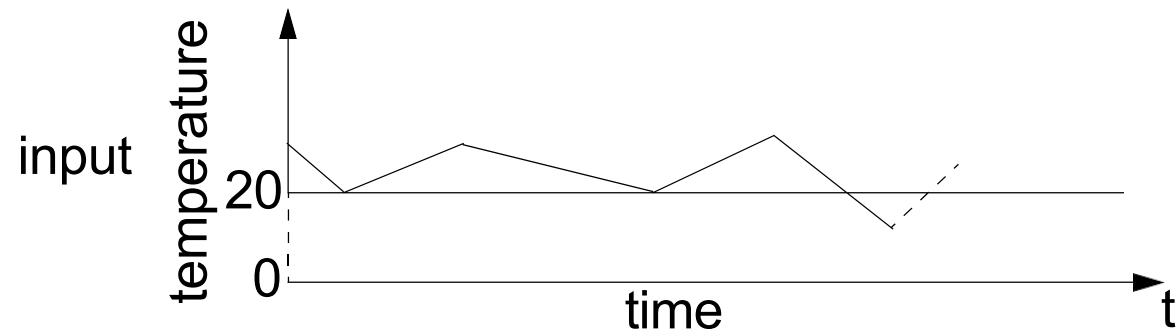
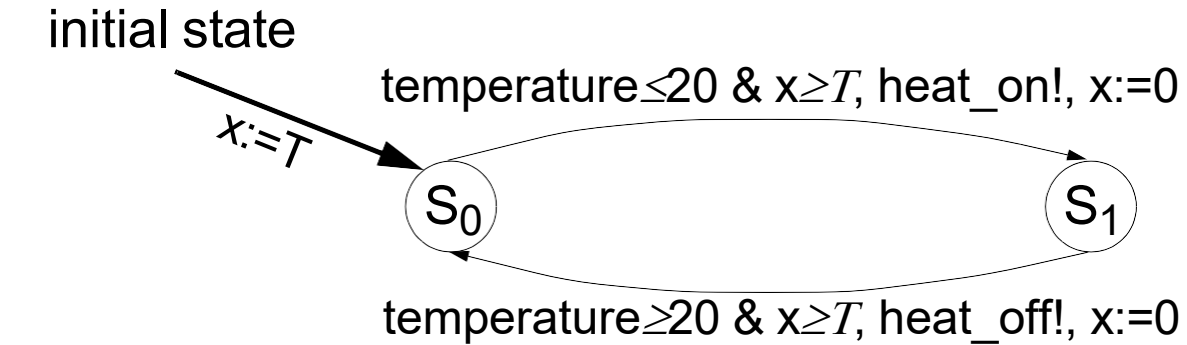
Example: Thermostat with Timed Automata

- Clock: x
- Input event: {temperature}
- Outputs: {heat_on, heat_off}
- States: $\{S_0, S_1\}$
 - S_0 : system cools (heating off)
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- We use a single temperature threshold - the desired level of 20° ;
To avoid chattering, the heater remains on/off for a minimum required time T .

Example: Thermostat with Timed Automata

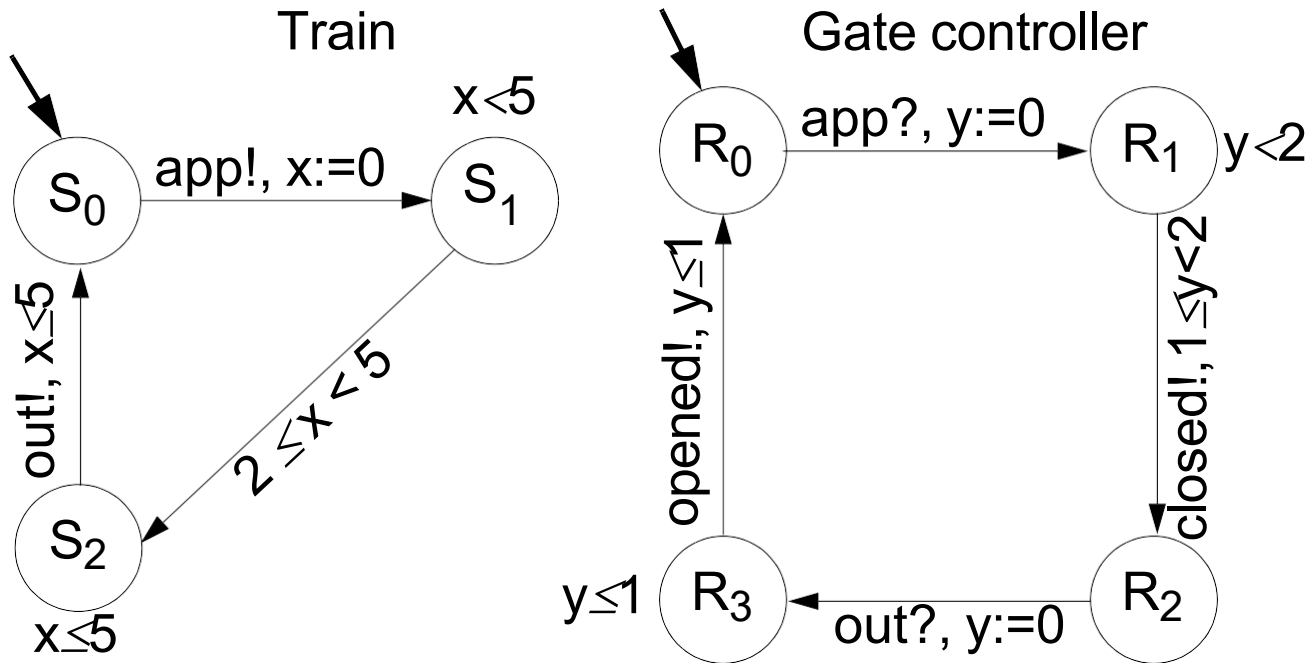


Example: Gate Control System

Specification:

- ❑ When the train approaches, it sends signal *app* at least 2 minutes before it enters the crossing; after leaving the crossing it sends signal *out*; it leaves the crossing maximum 5 minutes after signalling *app*.
- ❑ When the controller gets signal *app* it closes the gate, which takes at least 1 minute, but less than 2; then it waits for signal *out*; when *out* arrives it opens the gate within maximum 1 minute.

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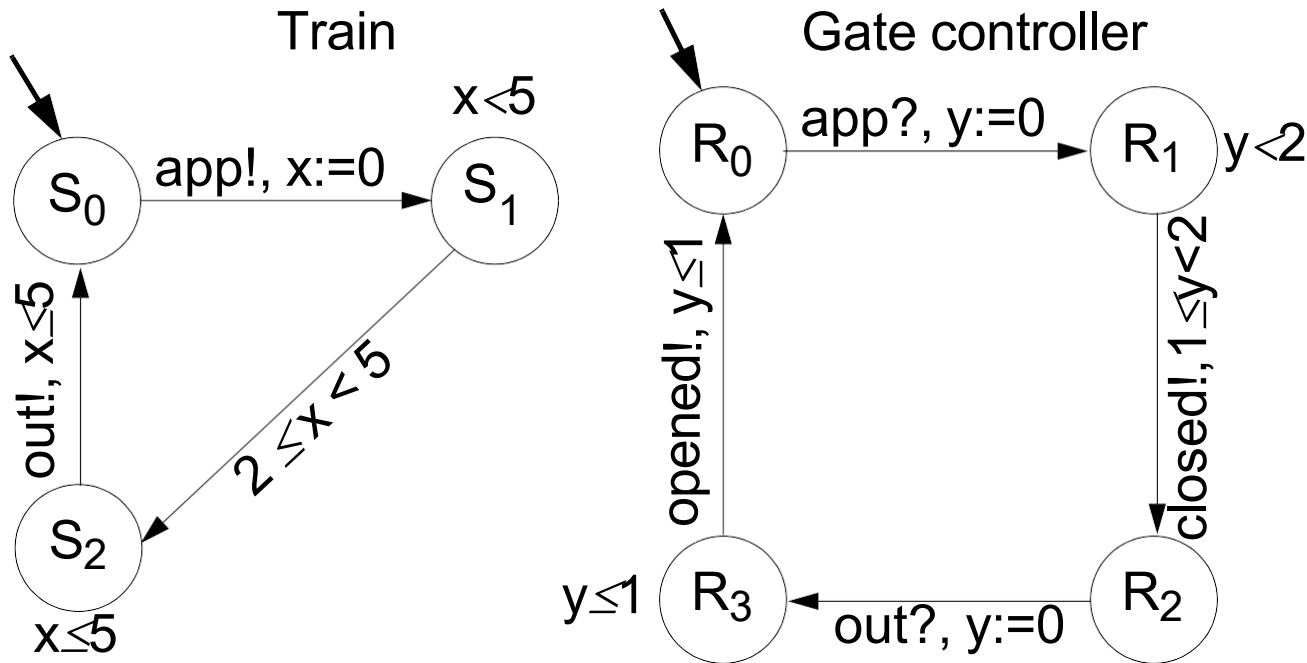


x, y: clocks

S₀: train far away
S₁: train approaching
S₂: train crossing

R₀: controller waiting
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R₂: gate is down
R₃: train has left

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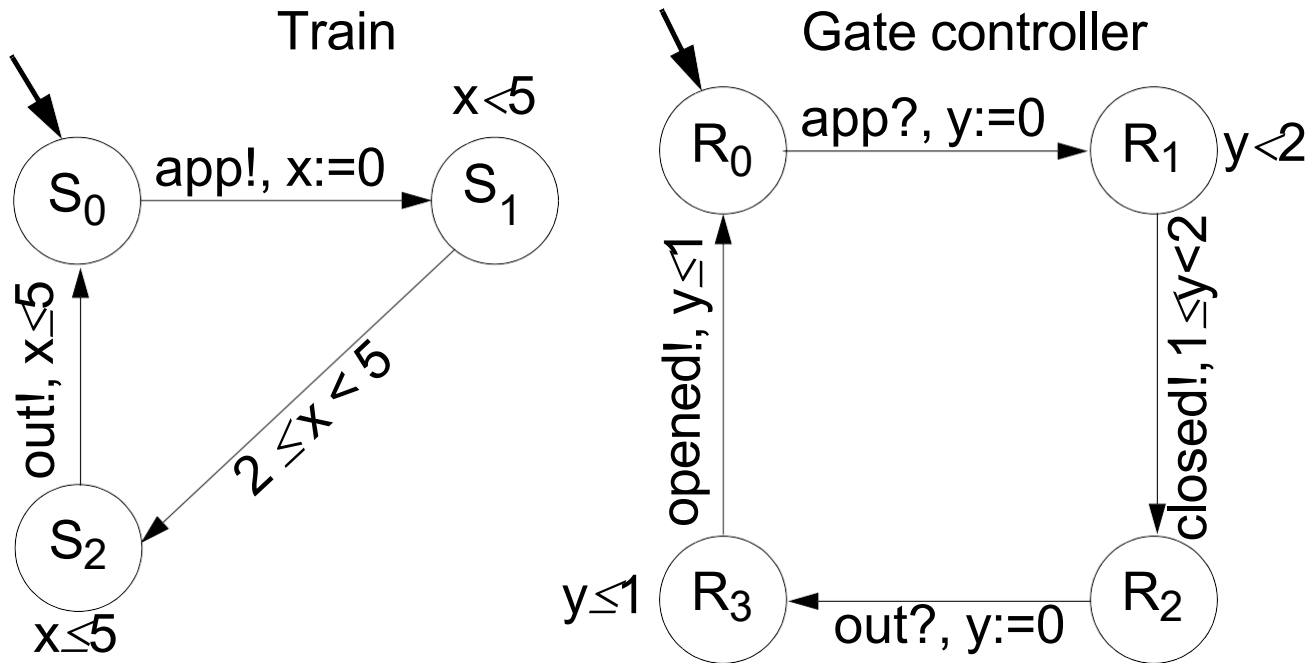
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- Once the above model is realised, one can formally verify (e.g. using model checking tools) properties such as: the train will only be in state S₂ (crossing) when, simultaneously, the gate is in R₂ (down).

Timed Automata: Rules/Properties



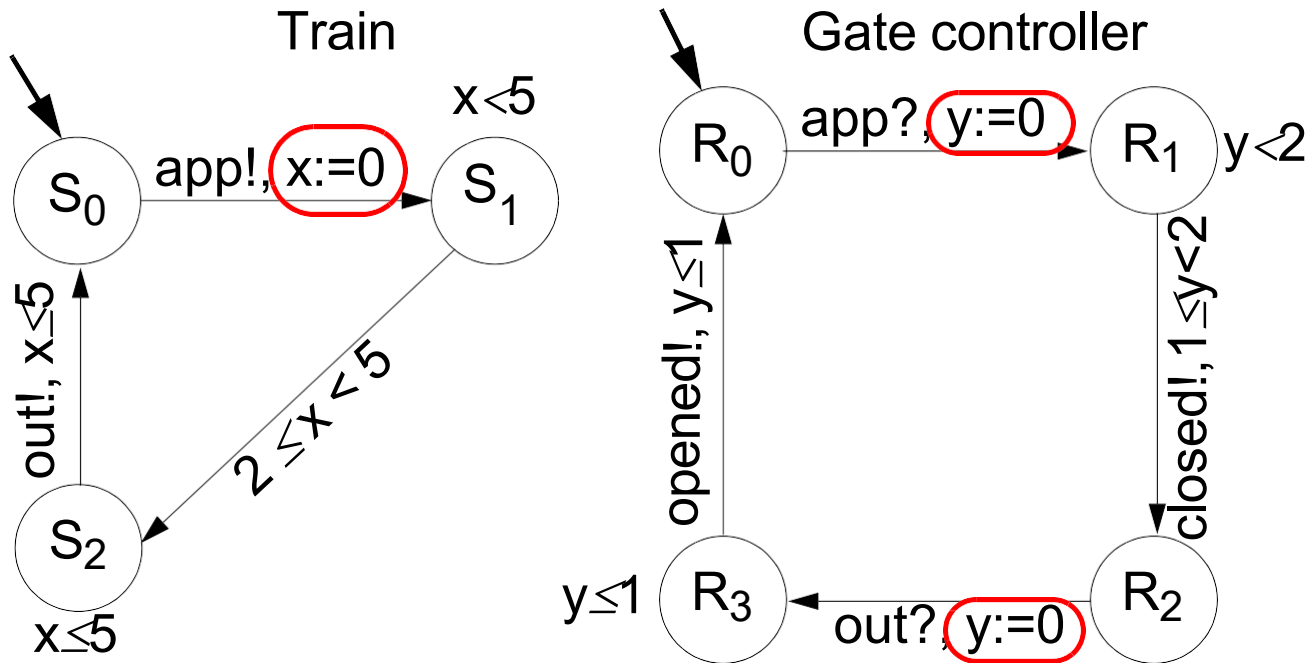
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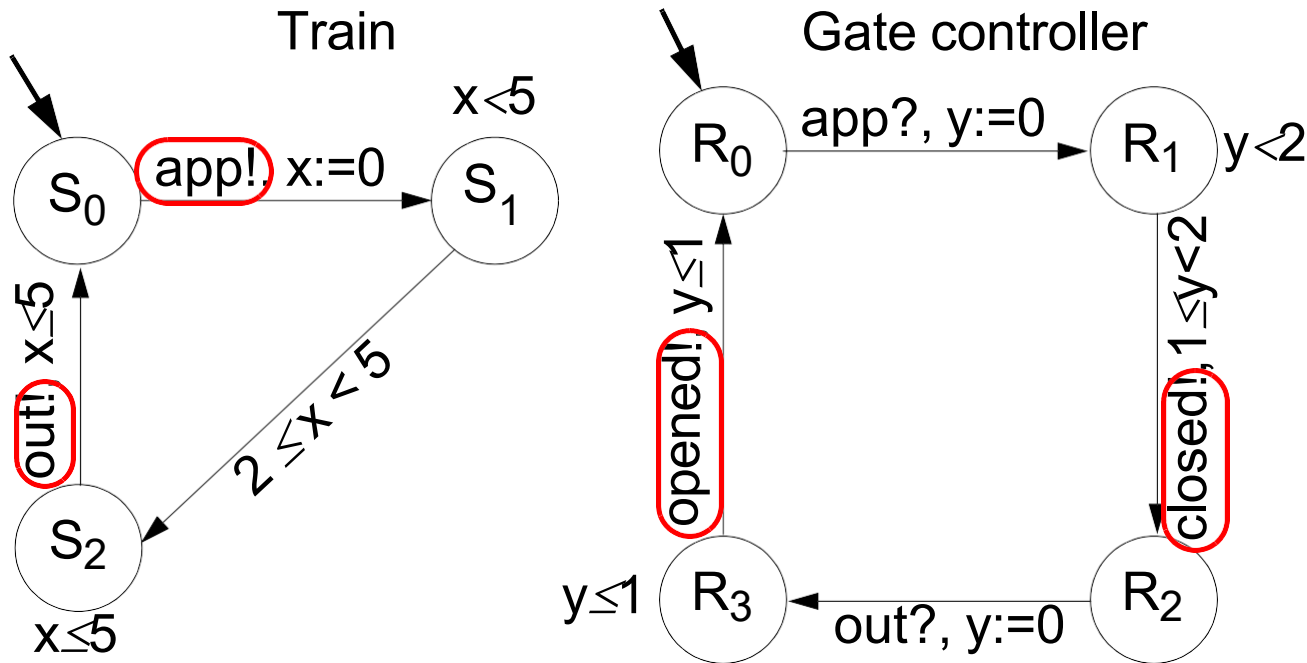
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- When a transition occurs clocks can be reset.

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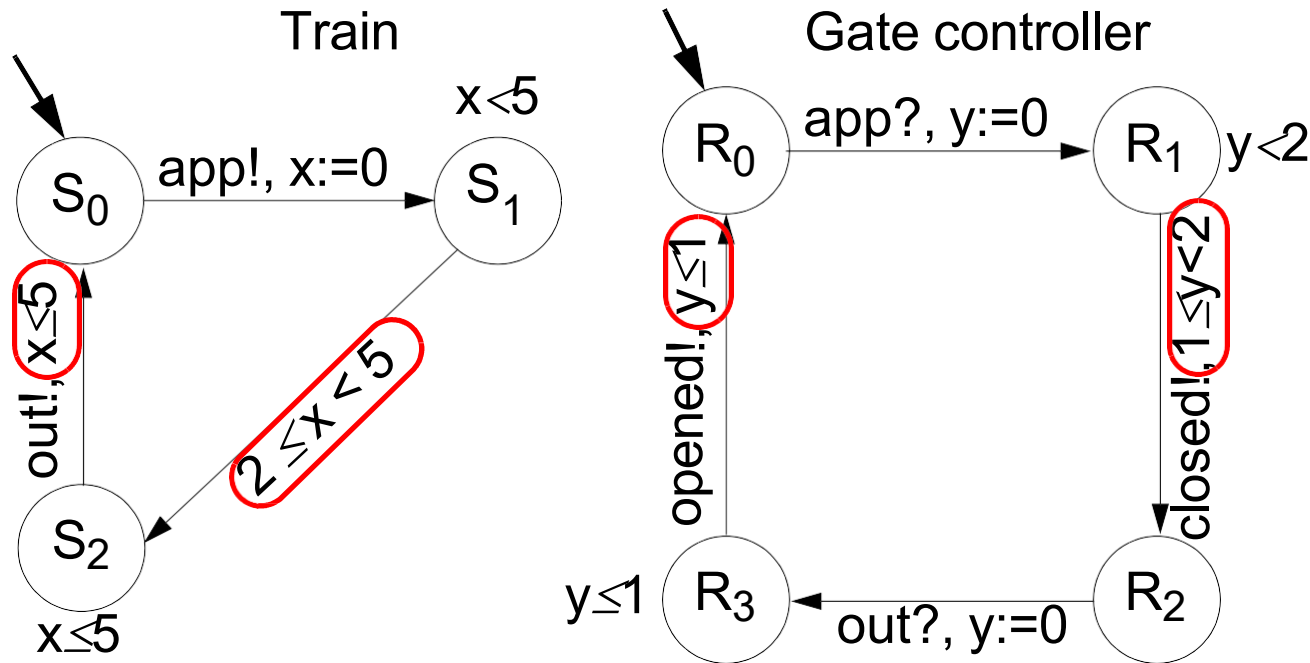
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- Transitions are instantaneous; time elapses when the automaton is in a state.
- When a transition occurs clocks can be reset.
- Time passes at the same rate for all clocks.
- When a transition occurs, signals (events) can be generated.

Timed Automata: Rules/Properties



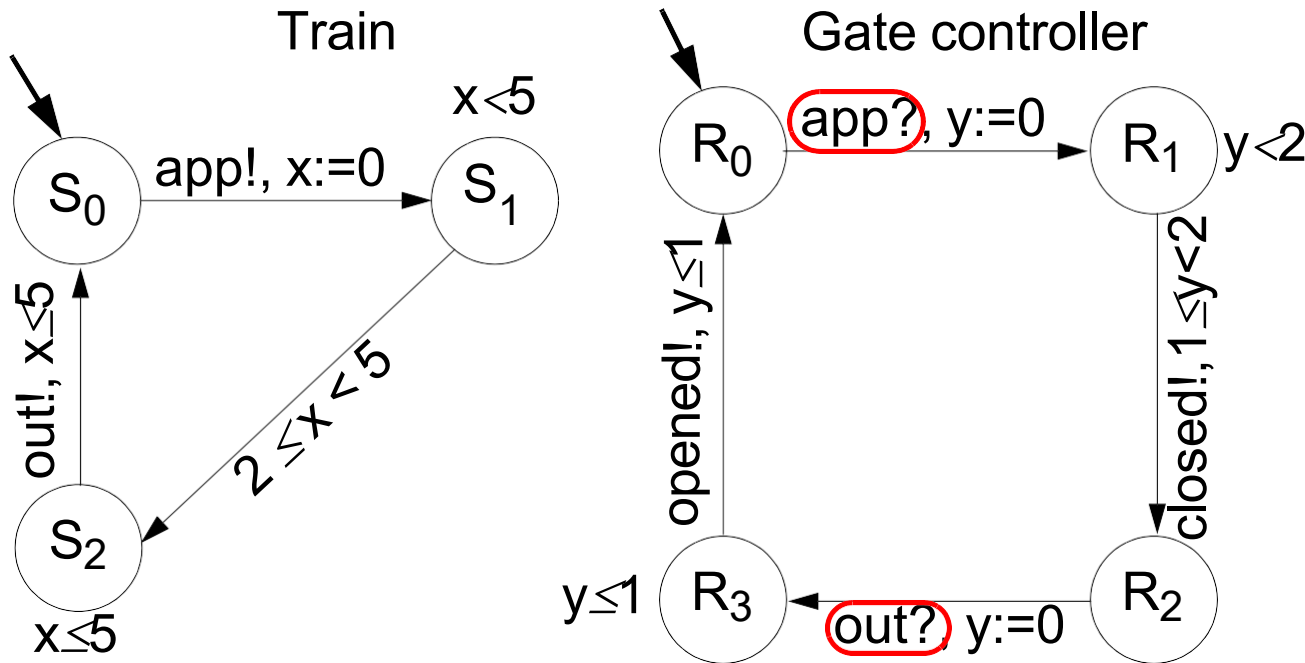
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- Transitions can have associated guards expressed as conditions on clock values; the transition can be taken only if the current values of the clocks satisfy the guard.

Timed Automata: Rules/Properties



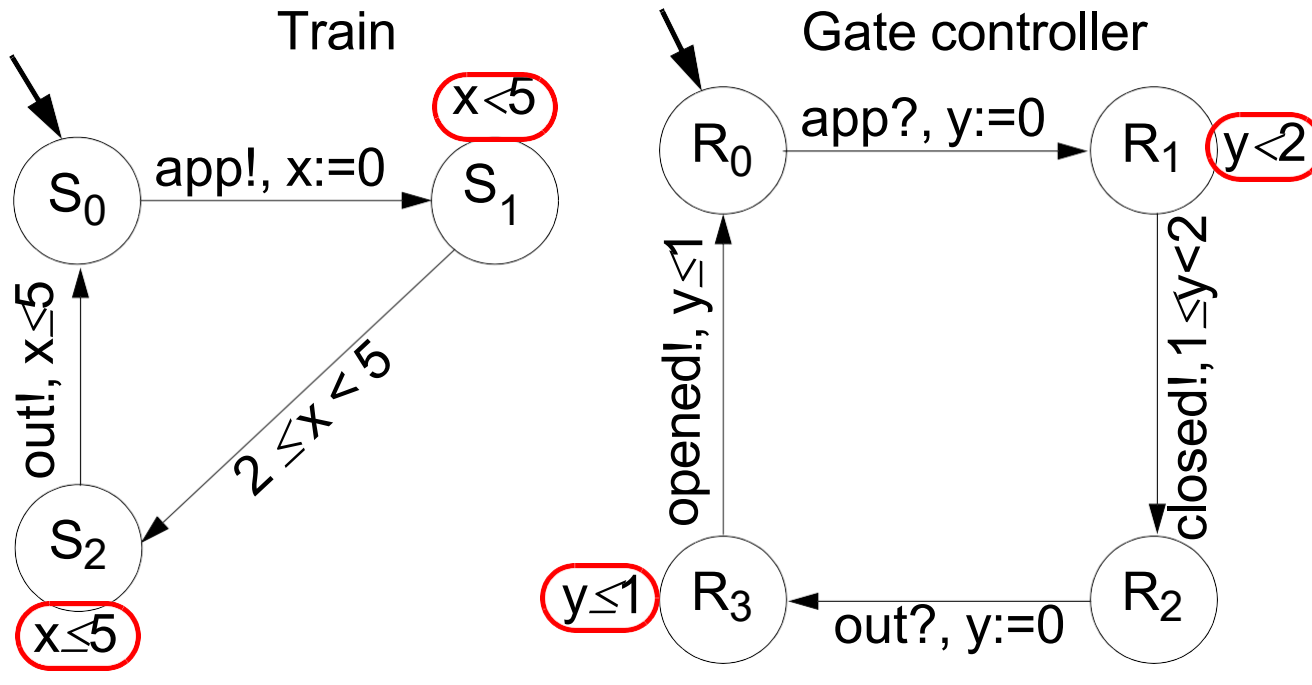
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- Transitions can have input signals (events) associated; when the signal arrives and the associated guard is satisfied, the transition will be taken.

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- Transitions can have input signals (events) associated; when the signal arrives and the associated guard is satisfied, the transition will be taken.
- States can have associated invariants, expressed as conditions on the clocks; the automaton can stay in that state as long as the invariant is true.

Timed Automata: Rules/Properties

- Like FSMs, timed automata can be extended with variables.
 - Actions on variables can be associated to transitions.
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- Like FSMs, timed automata can be extended with variables.
 - Actions on variables can be associated to transitions.
 - Guards expressed as conditions on the variables can be associated to transitions
- Timed automata are, by definition, *infinite state models*: At any time moment, the state of the system is defined not only by the actual state in the state machine (e.g. S_0 , S_1 , etc.), but also by the current values of the clocks!

However, for verification, timed automata admit *finite state representations* (by exploiting equivalence relations on certain portions of the state space)!

- Model checking techniques can be used to prove properties of timed automata.
- The state explosion problem is more severe than for synchronous concurrent FSMs!

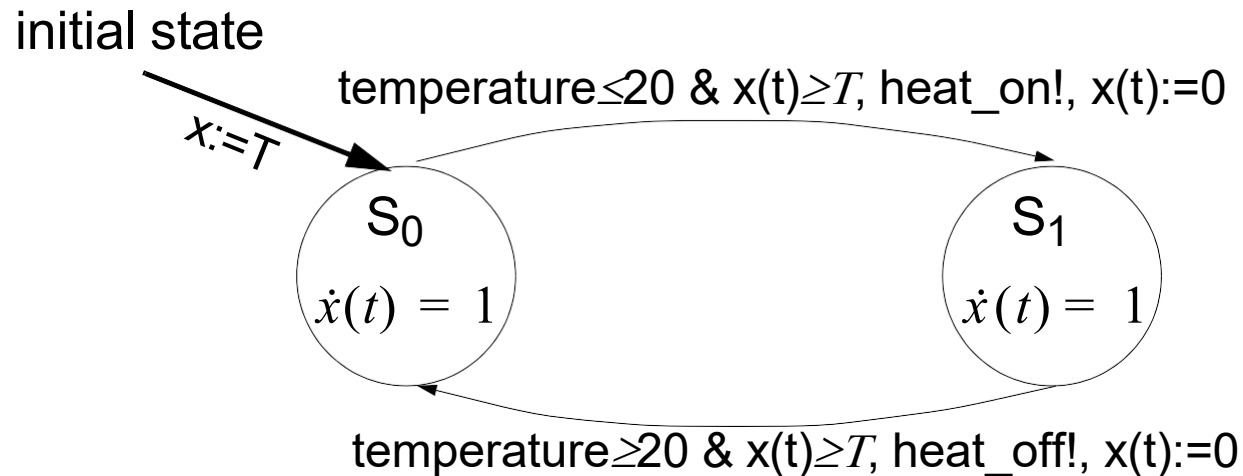
Hybrid Automata

- Timed automata are FSMs with addition of clocks.
- A clock is a continuous variable whose value can be described by the differential equation: $\dot{x}(t) = 1$.

Hybrid Automata

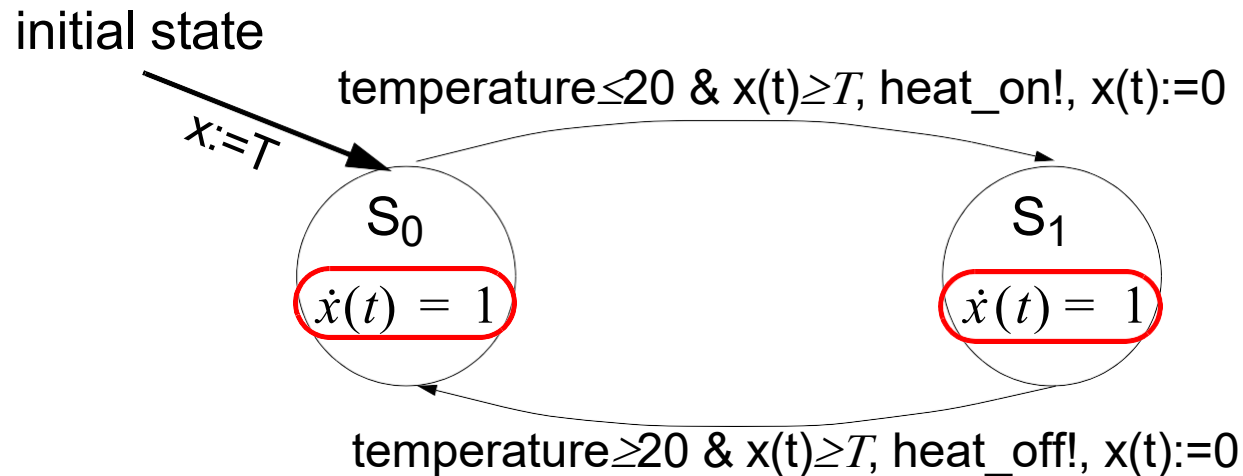
- Timed automata are FSMs with addition of clocks.
- A clock is a continuous variable whose value can be described by the differential equation: $\dot{x}(t) = 1$.
- Timed automata are, in fact, the simplest form of *hybrid automata*.
- Hybrid automata are FSMs combined with a finite set of continuous variables whose values are described by a set of ordinary differential equations.

The Thermostat as a Hybrid Automaton



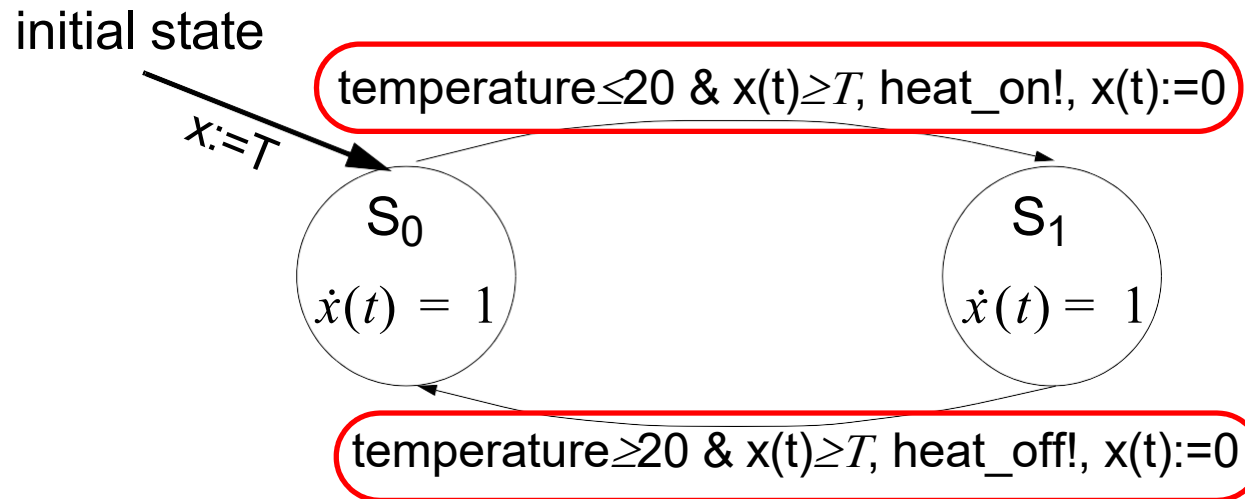
- In the above model, the equation describing the continuous variable representing clock x is made explicit. This model is completely equivalent with the one based on Timed Automata.

Hybrid Automata: Rules/Properties



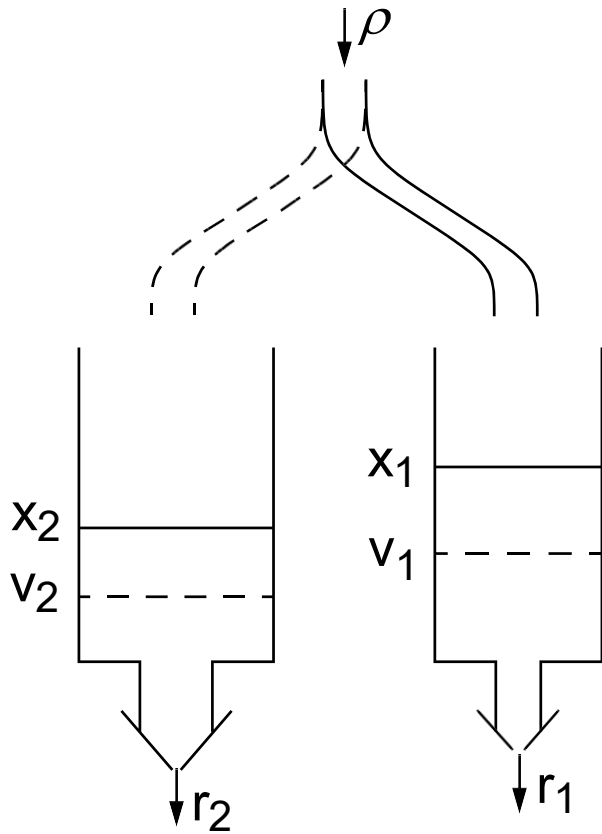
- Hybrid Automata associate with each state of an FSM a dynamic behavior.
- The dynamic behavior in each state is specified by a *state refinement*; a state refinement describes the dynamics of the outputs as a function of the inputs.
- State refinements are specified as ordinary *differential equations*.

Hybrid Automata: Rules/Properties



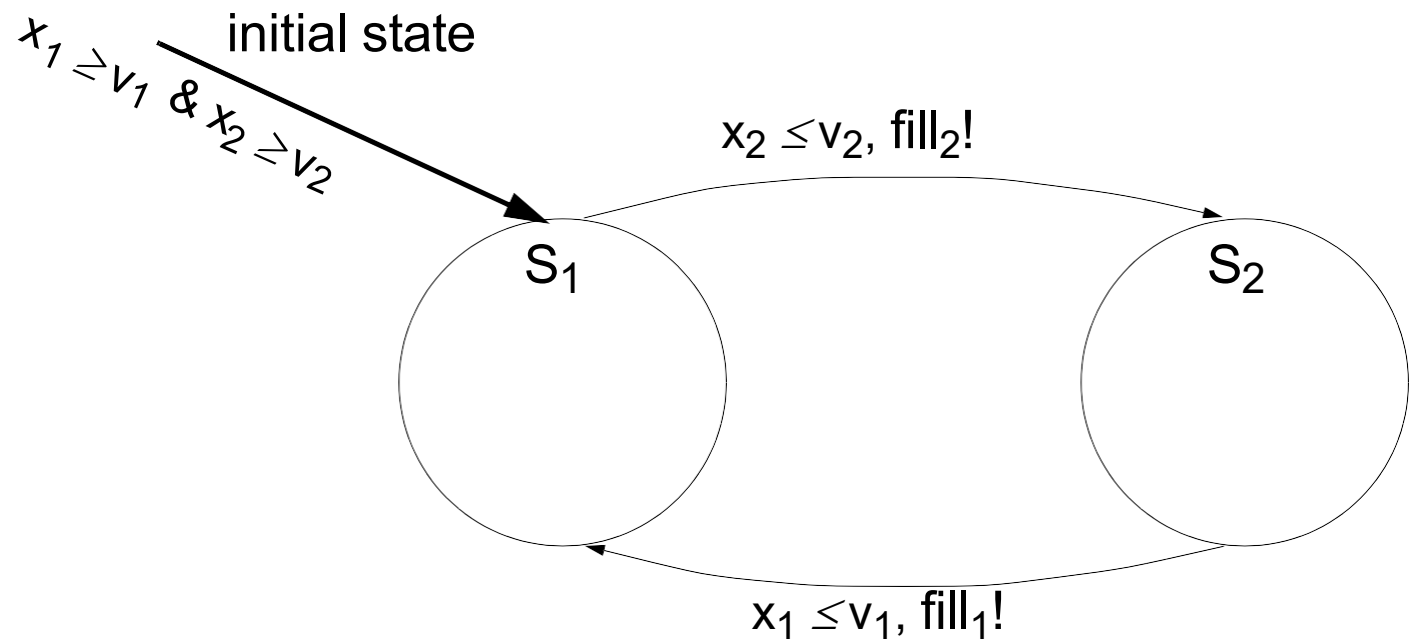
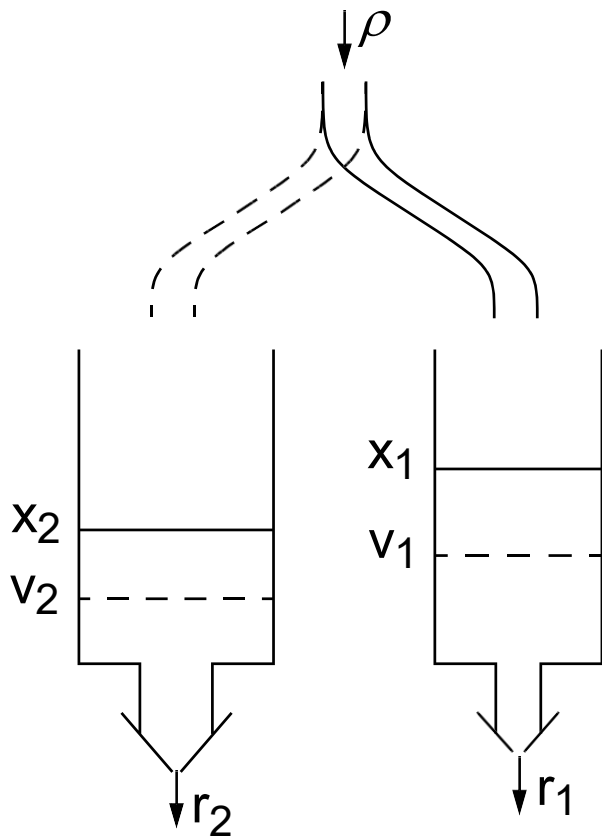
- Hybrid Automata associate with each state of an FSM a dynamic behavior.
- The dynamic behavior in each state is specified by a *state refinement*; a state refinement describes the dynamics of the outputs as a function of the inputs.
- State refinements are specified as ordinary *differential equations*.
- Transitions can have associated guards, assignments to variables, outputs...
- Hybrid automata are extremely strong in their expressive power; they combine discrete and continuous behavior in one single model.

Example: Water Tank



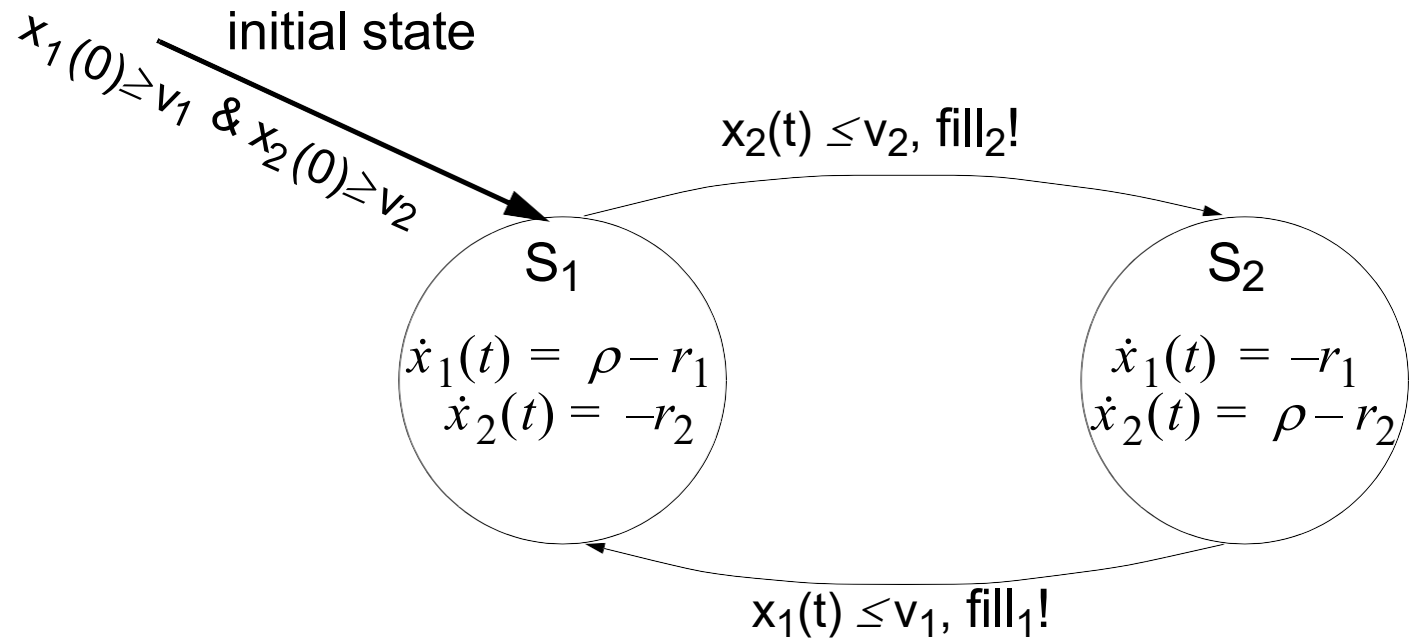
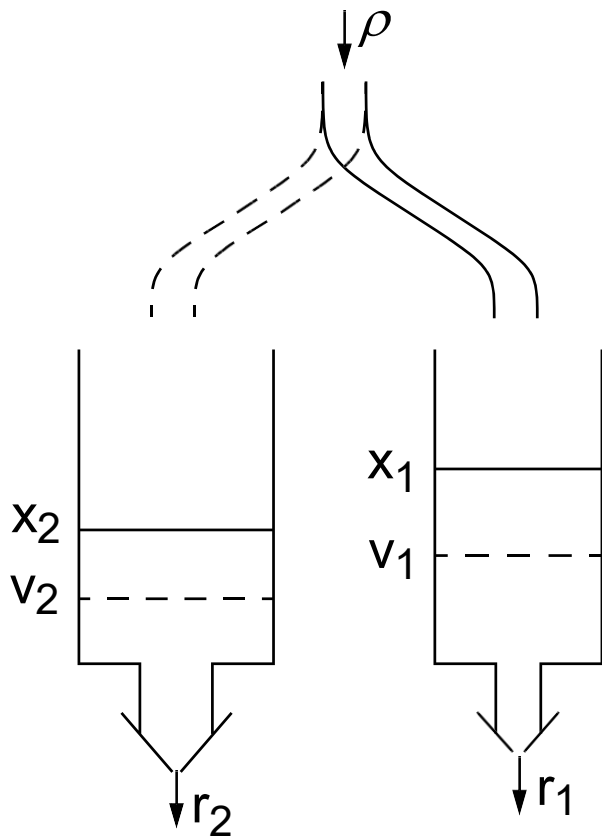
- Each tank is leaking at constant rate (r_1, r_2).
- Water is added at a constant rate ρ .
- One tank is filled at a time; filling switches from one tank to the other in zero time.
- The goal is to keep the water *volume* above v_1 and v_2 respectively.
- The current water volume is x_1, x_2 .

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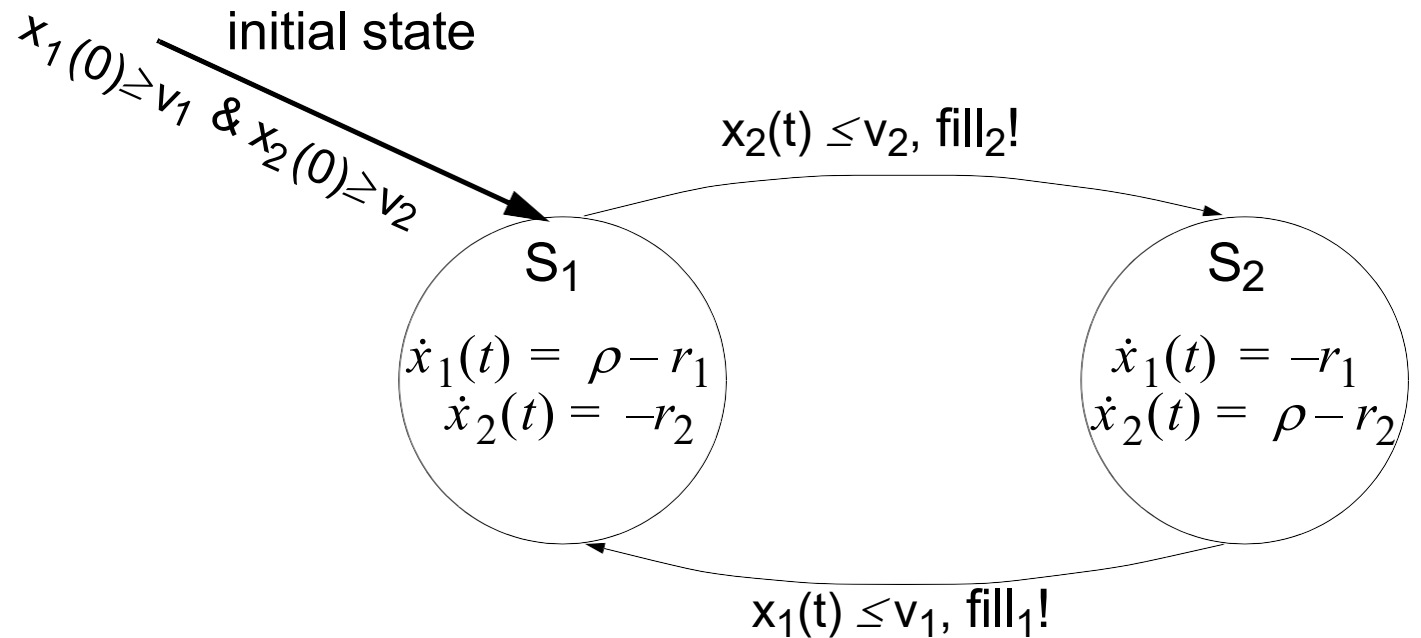
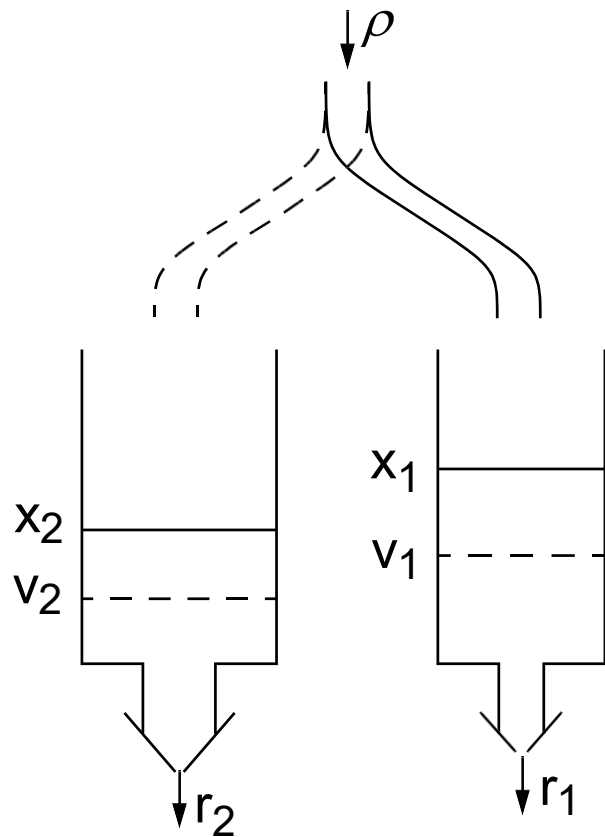
- Inputs: $\{x_1, x_2\}$
- Outputs: $\{\text{fill}_1, \text{fill}_2\}$
- States: $\{S_1, S_2\}$
 - S_1 : tank 1 is filled
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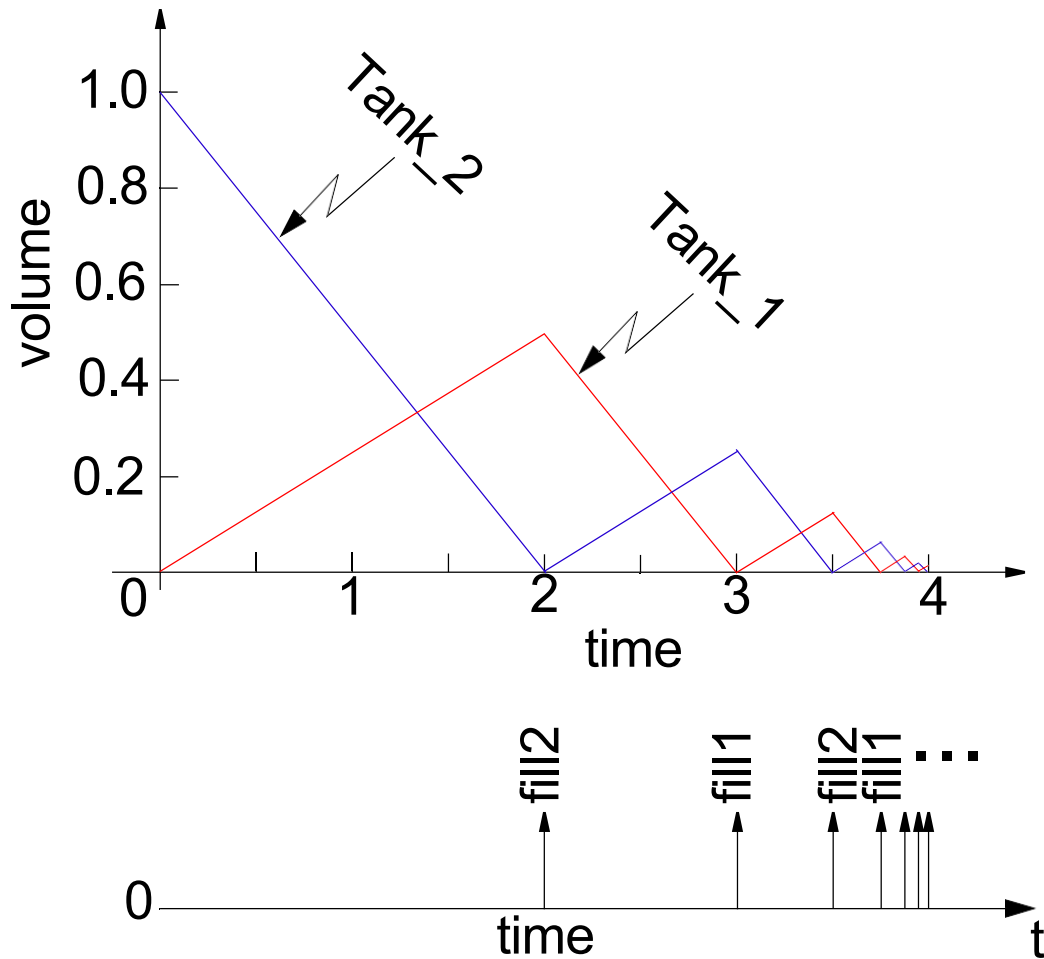
- The system might reach a situation (when water level is very close to the target) in which the number of switches per time unit is continuously increasing (Zeno system).

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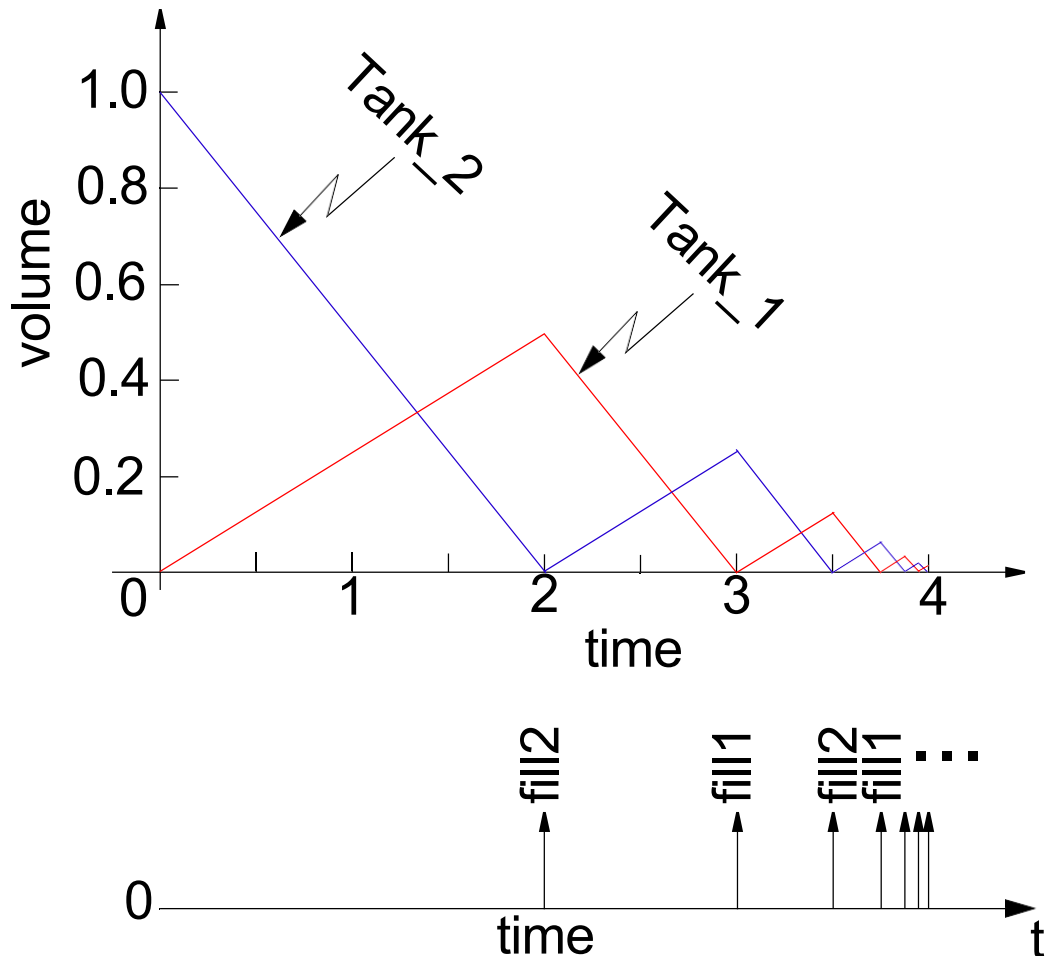
Example: Water Tank

This is a simulation considering:

- Leaking rates: $r_1=r_2=0.5$
 - Water inflow rate: $\rho=0.75$
 - Keep the water above $v_1=v_2=0$
 - Initial level: $x_1(0)=0, x_2(0)=1$
- Since ρ is too small, both tanks will, eventually, become empty.
 - As the tanks come close to the 0 level, the number of switches per time unit increases.



Example: Water Tank

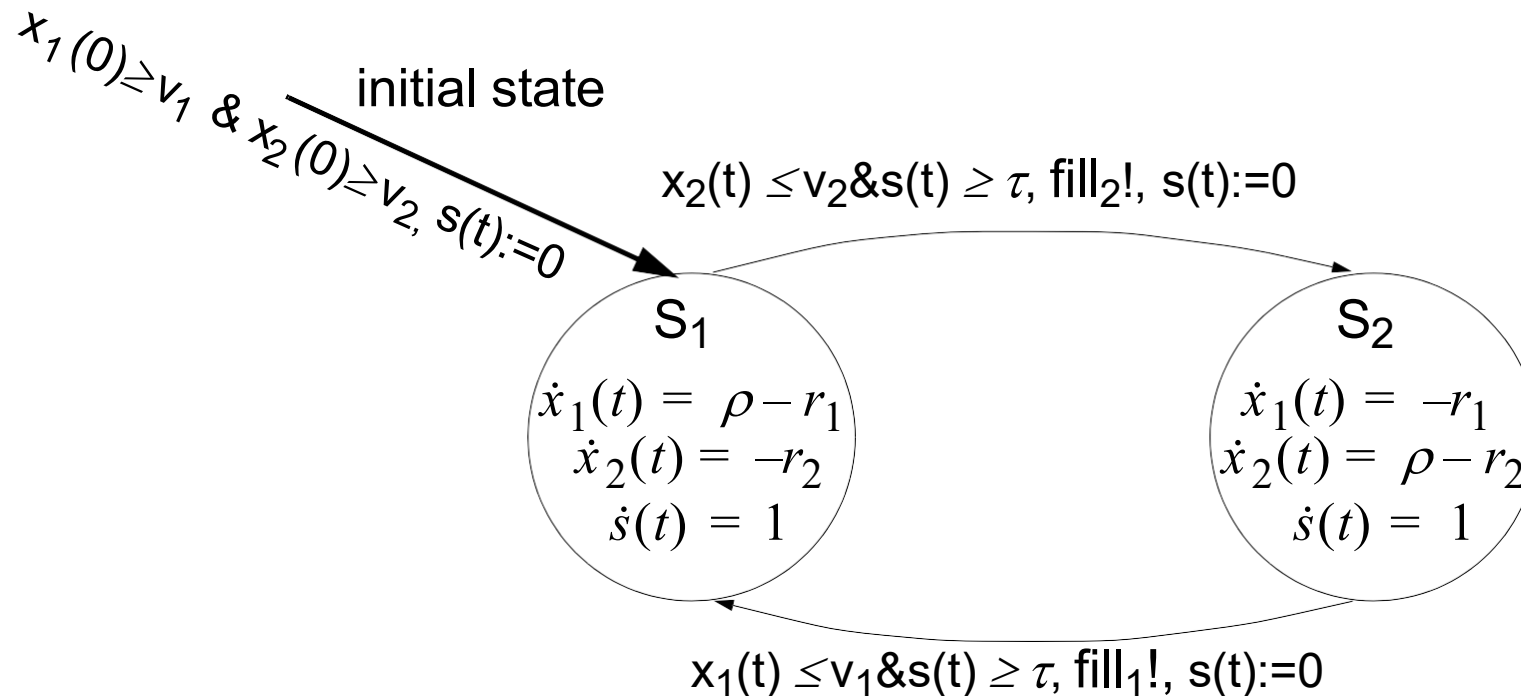


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- As the tanks come close to the 0 level, the number of switches per time unit increases.
- With such a Hybrid Automata model, one can use formal verification to answer questions like: *will tank_1 become empty before time 7?* You cannot ask this with a simple timed automata model (since the flow equations are not part of the model).

Example: Water Tank

- In order to avoid Zeno behaviour one possible solution is to allow switches only after the system spent a minimum amount of time τ in a state.



Hybrid Automata: Final Comments

- Hybrid automata models can be used for simulation and formal verification.
- Hybrid automata, like timed automata, are, by definition, *infinite state models*. However, they admit a *finite state representation* (by exploiting equivalence relations on certain portions of the state space)!
 - Model checking techniques can be used to prove properties of hybrid automata.
 - The state explosion problem is more severe than for timed automata!
- Available frameworks/tools: Ptolemy II, HyTech.

What Modeling Approach to Choose?

- It depends on the characteristics of the system:
 - control or data flow dominated;
 - synchronous or asynchronous; centralised or distributed;
 - how large?
 - what aspects related to timing are we interested in?

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- It depends on what tools you have available and which approach you (or your company or your boss!) prefer.

What Modeling Approach to Choose?

- Don't use the “strongest”! Go for exactly that expressive power you need; not more!
 - Large expressive power:
 - Can specify “anything”.
 - No formal reasoning possible (or extremely complex).
 - Limited expressive power, based on well chosen computation model:
 - Only particular systems can be specified.
 - Formal reasoning is possible.
 - Efficient implementation

Modeling Languages

- The choice of a modeling language is, to a large extent, connected to the choice of the modeling approach.

This, because certain modeling languages are strongly connected to a particular model of computation:

- ❑ Communicating asynchronous state machines: SDL, Lotos
- ❑ Synchronous FSM systems: Esterel, StateCharts;
- ❑ Dataflow computation: Matlab, Lustre, Silage
- ❑ Discrete event: SystemC, VHDL, Verilog